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Workout — FDM

Mandatory exercises

1. Consider the following advection equation (where the subscripts denote partial differentiation with respect to t and x, respectively):

$$\begin{cases} u_t = u_x, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t), & t > 0, \\ u(x,0) = f(x), & 0 < x < 1, \end{cases}$$

which is approximated by

$$\begin{cases} \frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}, & j = 0, 1, \ldots, N-1, \quad n = 0, 1, \ldots, \\ u_j^n = u_{j+N}^n, & \text{for all j}, \quad n = 0, 1, \ldots, \\ u_j^0 = f(x_j), & j = 0, 1, \ldots, N-1, \end{cases}$$

   (a) Derive the local truncation error of this approximation.

   (b) What is the order of accuracy of the approximation above?

   (c) What conclusion can you draw about consistency based on the local truncation error?

2. For the same approximation as above, use the von Neumann approach to analyze the stability. What is the stability condition?

   (a) For the same approximation as above, use the von Neumann approach to analyze the stability. What is the stability condition?

   (b) What conclusion can you draw about convergence, based on the combined results of Exercise 1 and Exercise 2a?

Non-mandatory exercises

3. In general, a centered, second order finite difference approximation of a second derivative u'', can be written as $(u_{i+1} - 2u_i + u_{i-1})/h^2$, where h is the step size.

   (a) Use this idea to formulate a finite difference approximation of the Laplace equation $u_{xx} + u_{yy} = 0$. Denote the step sizes by Δx and Δy, respectively.

(b) Suppose that this approximation is applied to the Laplace equation on the unit square, i.e., $0 \leq x \leq 1$, $0 \leq y \leq 1$, with given boundary values. This will result in an algebraic system of equations. Express this system on matrix-vector form.

(c) The coeffiecient matrix has a special structure. What will be the complexity for LU factorization of this matrix, if you take advantage of the structure? Your answer can refer to known complexity results from the course Scientific Computing I (*Beräkningsvetenskap I*).

4. One way to solve PDEs numerically is to use the “method of lines”. This means to first discretize in space but not in time (a so called *semi-discretization*). The result will be a system of ODE. This system can then be solved with a numerical ODE solver.

   Now, apply this idea to the heat equation. For the space discretization, use the same approximation of the second order derivative as in Exercise 3 above. Formulate the corresponding ODE system and apply the Trapezoid method to it to get a fully discrete approximation. You do not need to carry out any computations, only to express the approximation as a formula. Do you recognize the resulting method? What is its name according to the course text book, Chapter 9?

5. Suppose that you are going to solve the heat equation $u_t = cu_{xx}$ with a finite difference method. First you discretize in space, using the second order centered finite difference approximation $(u_{j-1} - 2u_j + u_{j+1})/(\Delta x)^2$ of the second derivative. Then you want to choose between Explicit Euler and Implicit Euler for advancing the solution in time. You want to use the time marching method (Explicit or Implicit Euler) that gives the shortest execution time for a given time step Δt. Which method will you choose? Is there any benefit in using the more expensive method?