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Scientific Computing III

Workout — FEM

Mandatory exercises

1. Consider the following boundary value problem

$$\begin{aligned} -u'' &= 10\sin(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned}$$

Define the piecewise linear hat functions $\{\phi_i(x)\}_{i=1}^{n-1}$ by $\phi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, and the linear space $V_h = \{v \in C([0,1]) : v(0) = v(1) = 0,\ v(x)$ piecewise linear on $[x_i, x_{i+1}]$, $i = 0,\ldots,n-1\}$, where $x_i = ih$, $h = 1/n$. Then $V_h = span(\{\phi_i(x)\}_{i=1}^{n-1})$ and every function $v_h \in V_h$ can be written as a linear combination of the hat functions. In particular, $u(x) \approx u_h = \sum_{i=1}^{n-1} c_i\phi_i(x)$ for some coefficients c_i. Define the finite element method using these basis functions. Use the case with $n = 4$.

   (a) Plot the base functions.

   (b) Derive the linear system $A\vec{c} = \vec{b}$ that gives the coefficients c_i in the finite element solution.

   (c) (If you have time left) Solve the linear system, plot the finite element solution u_h and compare with the analytical solution, e.g., by plotting both.

Non-mandatory exercises

2. The stationary heat equation for a metal rod with one end at a fixed temperature, a constant heat flux at the other end, and a heat source function $f(x)$ is given by

$$\begin{cases} -u''(x) = f(x), \quad 0 < x < 1, \\ u(0) = 0, \\ u'(1) = 1. \end{cases}$$

Introduce a uniform grid $x_j = jh$, $j = 0,\ldots,n$, where $h = 1/n$. Define the piecewise linear hat functions $\{\phi_i(x)\}_{i=1}^{n}$ and the linear space $V_h = \{v \in C([0,1]) : v(0) = 0 \mid v(x)$ is linear on $[x_j,\ x_{j+1}],\ j = 0,\ldots,n-1\}$. Then $V_h = span(\{\phi_i(x)\}_{i=1}^{n})$. Plot the base functions. Derive the finite element method using linear hat functions as your basis functions and compute $u(x) \approx u_h = \sum_{i=1}^{n} c_i \phi_i(x)$. Give your final result as a linear system of equations. In particular, be careful deriving the a_{nn} element in the stiffness matrix and the b_n element in the right hand side vector.

3. Consider the following boundary value problem

$$\begin{aligned} -u'' + a(x)u &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u(1) &= 0, \end{aligned}$$

where $a(x)$ is a known function. What complications in the derivation of the stiffness matrix does this result in and how can we solve these complications. Derive the linear system of equations for the finite element solution using the linear hat functions on an uniform grid.