Workout — FEM

Non-mandatory exercises

2. You are going to solve the PDE

$$\begin{cases} \Delta u(x,y) &= 0, & (x,y) \in \Omega, \\ u(x,y) &= g(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

for two different domains Ω. In each case, you can choose between using the finite difference method and the finite element method. Make your choice and motivate it by pointing out the advantages or disadvantages of the respective methods.

(a) Ω is the unit square.

If the square is discretized with an orthogonal grid, both methods will produce linear systems of equations of equal size that can be solved directly or iteratively. Both methods also have the same accuracy. For FDM, using the second derivative approximation from earlier in the course, we get $\tau(x,y) = \mathcal{O}\left((\Delta x)^2 + (\Delta y)^2\right)$. According to Theorem 6 in Section 3.4.5 of the FEM compendium, the error of the FEM solution is $\mathcal{O}\left(h^2\right)$, where h is the step size in both directions. However, FDM requires less operations to construct the coefficient matrix, making it the better choice in this case.

Depending on the PDE, it may be wise to use FEM with an anisotropic mesh. In this way, it is possible to capture for example oscillations with a period that is a multiple of the step size. This will, however, add some non-zero entries to the coefficient matrix compared to a right-angled grid. It will also decrease the accuracy of the solution for a given h.

(b) Ω has the shape of a gear wheel (kugghjul).

FEM is the best alternative in this case, as the geometry is quite complex. While FDM is based on an orthogonal grid and would be much more complicated if extended to the more general case, the complexity of FEM doesn't increase much for irregular grids and domains. Moreover, the size and shape of the finite elements can be adapted to the shape of the domain, so that a good approximation can be obtained using a mesh that is coarse in easy parts of the domain and finer in the more difficult areas. On average, the mesh may then be fairly coarse, yielding a relatively small system of equations. To reach a comparable accuracy with FDM, a very fine mesh is needed, adding computational complexity to the problem.

3. Consider the boundary value problem in exercise **??**. If $a = a(x)$ is a function of x, what effects does that have on the computations?

Every $K_{i,j}$ needs to be computed individually, which adds a lot more computational complexity (or rather removes several possibilities for optimization) and the problem will be heavier to solve.

4. The stationary heat equation for a metal rod with one end at a fixed temperature, a constant heat flux at the other end, and a heat source function $f(x)$ is given by

$$\begin{cases} -u''(x) &= f(x), \quad 0 < x < 1, \\ u(0) &= 0, \\ u'(1) &= 1. \end{cases}$$

(a) Derive the weak formulation of the problem. The space

$$V^0 = \{v(x) \mid \; v(0) = 0, \; v \text{ is piecewise continuously differentiable on } 0 \le x \le 1\},$$

can be used both for the weak solution $u(x)$ and for the test functions $v(x)$.
Multiply with a test function:

$$-u''v = fv$$

Integrate:

$$-\int_0^1 u''v dx = \int_0^1 u'v' dx - [u'v]_0^1 = \int_0^1 u'v' dx + v(1) = \int_0^1 fv dx$$

$$\Rightarrow \int_0^1 u'v' dx = \int_0^1 fv dx - v(1)$$

That is, find $u \in V^0$ such that

$$\int_0^1 u'v' dx = \int_0^1 fv dx - v(1) \qquad \forall v \in V^0.$$

(b) Introduce a uniform grid $x_j = jh$, $j = 0, \dots, n$, where $h = 1/n$. Discretize the weak form of the PDE using the space

$$V_h^0 = \{v(x) \in V^0 \mid v(x) \text{ is linear on } [x_j, x_{j+1}], \; j = 0, \dots, n-1\},$$

and derive the finite element method using linear hat functions as your basis functions. Give your final result as a linear system of equations, where the matrix elements are given explicitly, but the right hand side may contain integrals with the function $f(x)$.
Hint: Make a figure of your hat functions in order to get *all* the integrals right.
The finite element method reads: Find $u_h \in V_h^0$ such that

$$\int_0^1 u_h'v' dx = \int_0^1 fv dx - v(1) \qquad \forall v \in V_h^0$$

To derive the corresponding system of equations, insert the approximated solution $u_h = \sum_{i=1}^n c_i\phi_i$, where $\phi_i = \phi_i(x)$ are linear hat functions, into the integrals:

$$\int_0^1 u_h'v' dx = \int_0^1 \sum_{i=1}^n c_i\phi_i'v' dx = \sum_{i=1}^n c_i \int_0^1 \phi_i'v' dx = \int_0^1 fv dx - v(1)$$

Note that the upper limit of the sum is now n, and not $n-1$ as before, as we have a Neumann condition at $x = 1$. Use ϕ_j, $j = 0, \dots, n-1$, as test functions and use the fact that every ϕ_j is 0 for all x not in $[x_j, x_{j+1}]$:

$$\begin{aligned}
\sum_{i=1}^{n} c_i \int_0^1 \phi_i' \phi_j' dx &= \sum_{i=j-1}^{j+1} c_i \int_0^1 \phi_i' \phi_j' dx \\
&= c_{j-1} \int_{x_{j-1}}^{x_j} \phi_{j-1}' \phi_j' dx + c_j \int_{x_{j-1}}^{x_{j+1}} \phi_j'^2 dx + c_{j+1} \int_{x_j}^{x_{j+1}} \phi_{j+1}' \phi_j' dx \\
&= \int_0^1 f \phi_j dx - \phi_j(1)
\end{aligned}$$

The function ϕ_j:

$$\phi_j = \begin{cases} \frac{x - x_{j-1}}{h} & \text{for } x \in [x_{j-1}, x_j) \\ \frac{x_{j+1} - x}{h} & \text{for } x \in [x_j, x_{j+1}) \\ 0 & \text{elsewhere} \end{cases}$$

Inserting this into the integrals, and using the fact that $x_j - x_{j-1} = h$ for all j, we get:

$$\begin{aligned}
& c_{j-1} \int_{x_{j-1}}^{x_j} \left(-\frac{1}{h^2}\right) dx + c_j \int_{x_{j-1}}^{x_{j+1}} \frac{1}{h^2} dx + c_{j+1} \int_{x_j}^{x_{j+1}} \left(-\frac{1}{h^2}\right) dx \\
&= -\frac{1}{h} c_{j-1} + \frac{2}{h} c_j - \frac{1}{h} c_{j+1} \\
&= \frac{1}{h} \int_{x_{j-1}}^{x_j} (x - x_{j-1}) f dx + \frac{1}{h} \int_{x_j}^{x_{j+1}} (x_{j+1} - x) f dx - \phi_j(1)
\end{aligned}$$

Remember, $\phi_j(1)$ is non-zero only for $j = n$, where $\phi_n(1) = \phi_n(x_n) = 1$. The resulting system of equations can be written as:

$$\begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & 0 & \dots & 0 \\ -\frac{1}{h} & & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & \frac{2}{h} & -\frac{1}{h} \\ 0 & \dots & 0 & -\frac{1}{h} & \frac{1}{h} \end{pmatrix} \begin{pmatrix} c_1 \\ \\ \vdots \\ \\ c_n \end{pmatrix} = \begin{pmatrix} \int_{x_0}^{x_2} f \phi_1 dx \\ \vdots \\ \int_{x_{n-2}}^{x_n} f \phi_{n-1} dx \\ \int_{x_{n-1}}^{x_n} f \phi_n dx - 1 \end{pmatrix}$$

The last row is special, as the integration interval doesn't include x_{n+1}.